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Title: PRELIMINARY RESULTS ON THE STABILITY OF SWITCHED POSITIVE LINEAR SYSTEMS

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Preliminary results on the stability of switched positive linear systems

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Abstract—**It was recently conjectured that the Hurwitz stability of a polytope of Metzler matrices is a necessary and sufficient condition for the stability of the associated switched linear system for arbitrary switching sequences. In this paper we show: (i) that this conjecture is true for a polytope constructed from a pair of second order Metzler matrices; (ii) that the conjecture is true for general polytopes of a finite number of second order Metzler matrices; and (iii) that the conjecture is in general false for higher order systems. The implications of our results, both for the design of switched positive linear systems, and for research directions that arise as a result of our work, are discussed toward the end of the paper.**

Key Words: Stability theory; Switched linear systems; Positive linear systems

I. INTRODUCTION

The theory of positive linear time-invariant systems has historically assumed a position of great importance in systems theory and has been applied in the study of a wide variety of dynamic systems [1], [2], [3], [4]. Recently, new application studies in communication systems [5], formation flying [6], and several other areas, have highlighted the importance of switched (hybrid) positive linear systems (PLS). While the main properties of positive LTI systems are well understood, many basic questions relating to switched PLS remain unanswered. The most important of these concerns their stability. In this paper we commence this study by presenting some basic results on the stability of switched PLS.

The layout of the paper is as follows. In section 2 we present the mathematical background and notation necessary to state the main results of the paper. Then in sections 3 and 4 we present necessary and sufficient conditions for the exponential stability of switched second order positive linear systems. In section 5 we show by means of an abstract construction that the results for derived in the preceeding sections do not generalise immediately to higher dimansional systems.

II. MATHEMATICAL PRELIMINARIES

In this section we present a number of preliminary results that are used later.

(i) Notation

Throughout, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}^n$ stands for the vector space of all n-tuples of real numbers and $\mathbb{R}^{m\times n}$ is the space of $m \times n$ matrices with real entries. For x in $\mathbb{R}^n$, x_i denotes the i^{th} component of x, and the notation $x \succ 0$ ($x \succeq 0$) means that $x_i > 0$ ($x_i \geq 0$) for $1 \leq i \leq n$. Similarly, for a matrix A in $\mathbb{R}^{n\times n}$, a_{ij} denotes the element in the (i, j) position of A, and $A \succ 0$ ($A \succeq 0$) means that $a_{ij} > 0 (a_{ij} \geq 0)$ for $1 \leq i, j \leq n$. $A \succ B$ ($A \succeq B$) means that $A - B \succ 0$ ($A - B \succeq 0$). We write A^T for the transpose of A. For P in $\mathbb{R}^{n\times n}$ the notation $P > 0$ means that the matrix P is positive definite. The spectral radius of a matrix A is the maximum modulus of the eigenvalues of A and is denoted by $\rho(A)$. Also we shall denote the maximal real part of any eigenvalue of A by $\mu(A)$. If $\mu(A) < 0$ (all the eigenvalues of A are in the open left half plane) A is said to be *Hurwitz*

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(ii) Positive LTI systems and Metzler matrices:

The LTI system

$$\Sigma_A : \dot{x}(t) = Ax(t), \quad x(0) = x_0$$

is said to be positive if $x_0 \succeq 0$ implies that $x(t) \succeq 0$ for all $t \geq 0$. Basically, if the system starts in the non-negative orthant of $\mathbb{R}^n$, it remain there for all time. See [3] for a description of the basic theory and several applications of positive linear systems. It is well-known that the system Σ_A is positive if and only if the off-diagonal entries of the matrix A are non-negative. Matrices of this form are known as Metzler matrices. If A is Metzler we can write $A = N - \alpha I$ for some non-negative N and a scalar $\alpha \geq 0$. Note that if the eigenvalues of N are $\lambda_1, \ldots \lambda_n$, then the eigenvalues of $N - \alpha I$ are $\lambda_1 - \alpha, \ldots \lambda_n - \alpha$. Thus the Metzler matrix $N - \alpha I$ is Hurwitz if and only if $\alpha > \rho(N)$.

The next result concerning positive combinations of Metzler Hurwitz matrices was pointed out in [7].

Lemma 2.1: Let A_1, A_2 be Metzler and Hurwitz. Then $A_1 + \gamma A_2$ is Hurwitz for all $\gamma > 0$ if and only if $A_1 + \gamma A_2$ is non-singular for all $\gamma > 0$.

(iii) Common quadratic Lyapunov functions

It is well known that the existence of a common quadratic Lyapunov function for the family of stable LTI systems

$$\Sigma_{A_i} : \dot{x} = A_i x \quad i \in \{1, \ldots, m\}$$

is sufficient to guarantee the exponential stability of the associated switching system

$$\Sigma_S : \dot{x} = A(t)x \quad A(t) \in \{A_1, \ldots, A_m\}. \tag{1}$$

Formally checking for the existence of a CQLF amounts to looking for a single positive definite matrix $P = P^T > 0$ in $\mathbb{R}^{n\times n}$ satisfying the m Lyapunov inequalities

$$A_i^T P + PA_i < 0 \quad i \in \{1, \ldots, m\}. \tag{2}$$

If such a P exists, then $V(x) = x^T Px$ defines a CQLF for the LTI systems Σ_{A_i}. While the existence of such a function is sufficient to assure the stability of the system (4), it is in general not necessary for stability [8]. Hence CQLF existence is in general a conservative way of establishing stability for switched linear systems. However, recent work has established a number of system classes for which this is not necessarily the case [9], [10]. The results in these papers relate the existence of a solution that is not globally attractive to a switched linear system to the Hurwitz stability of a polytope of matrices and are based on the following theorem.

Theorem 2.1: [11], [12] **:** Let $A_i \in \mathbb{R}^{n\times n}$, $i = \{1,2\}$, be general Hurwitz matrices. A sufficient condition for the existence of an unstable switching sequence for the system

$$\dot{x} = A(t)x,\ A(t) \in \{A_1, A_2\},$$

is that the matrix pencil $A_1 + \lambda A_2$ has an eigenvalue with a positive real part for some positive λ.

The relationship between the existence of a CQLF, the existence of an unbounded solution to a switched linear system and the Hurwitz stability of a polytope of matrices will play a crucial role in this paper.

Finally, we note that a defining characteristic of switched positive linear systems is that any trajectory originating in the positive orthant will remain there as time evolves. Consequently, to demonstrate the stability of such systems, one need not search for a common quadratic Lyapunov function, but rather the existence of a copositive Lyapunov function. Formally checking for the existence of a copositive CQLF amounts to looking for a single symmetric matrix P such that $x^T Px > 0$ for $x \in \mathbb{R}^n, x \succeq 0$, $x \neq 0$, satisfying the m Lyapunov inequalities

$$x^T(A_i^T P + PA_i)x^T < 0 \quad i \in \{1,\dots,m\},\ \forall x \succeq 0. \quad (3)$$

III. Pairs of Second Order Positive Linear Systems

In this section, we shall show that the conjecture in [13] is true for second order systems. To begin with we recall the result of [9] which described necessary and sufficient conditions for the existence of a CQLF for a pair of general second order LTI systems.

Theorem 3.1: Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz. Then a necessary and sufficient condition for Σ_{A_1}, Σ_{A_2} to have a CQLF is that the matrix products A_1A_2 and $A_1A_2^{-1}$ have no negative eigenvalues.

We next show that it is only necessary to check one of the products in the above Theorem if the individual systems $\Sigma_{A_1}, \Sigma_{A_2}$ are positive systems.

Lemma 3.1: Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz and Metzler. Then the product A_1A_2 has no negative eigenvalue.

Proof: First of all, as A_1, A_2 are both Hurwitz, the determinant of A_1A_2 must be positive. Secondly, a straightforward calculation shows that all of the diagonal entries of A_1A_2 must be positive. Hence the trace of A_1A_2 is also positive. It now follows easily that the product A_1A_2 cannot have any negative eigenvalues as claimed.

A straightforward combination of Theorem 3.1 and Lemma 3.1 yields the following result.

Theorem 3.2: Let $\Sigma_{A_1}, \Sigma_{A_2}$ be stable positive LTI systems with $A_1, A_2 \in \mathbb{R}^{2\times 2}$. Then:

(a) Σ_{A_1} and Σ_{A_2} have a CQLF;
(b) Σ_{A_1} and Σ_{A_2} have a common copositive quadratic Lyapunov function;
(c) Σ_S is globally uniformly exponentially stable;

if and only if the matrix product $A_1A_2^{-1}$ has no negative eigenvalues.

Proof :

(a) From Lemma 3.1 it follows that the matrix product A_1A_2 cannot have a negative eigenvalue. Hence, from Theorem 3.1, a necessary and sufficient condition for a CQLF for Σ_{A_1} and Σ_{A_2} is that $A_1A_2^{-1}$ has no negative eigenvalue.
(b) Since the fact that $A_1A_2^{-1}$ has no negative eigenvalue is necessary and sufficient for a CQLF for Σ_{A_1} and Σ_{A_2}, it follows that this condition is certainly sufficient for the existence of a copositive common quadratic Lyapunov function for this pair of positive LTI systems. Suppose now that $A_1A_2^{-1}$ has a negative eigenvalue. It follows that $A_1 + \gamma A_2$ has a real non-negative eigenvalue for some $\gamma_0 \in \mathbb{R}^+$. Since, $A_1 + \gamma_0 A_2 = N - \alpha_0 I$, where $N \succeq 0$, it follows that the eigenvector corresponding to this eigenvalue is the Perron eigenvector of N and consequently lies in the positive orthant [2]. It follows that a copositive Lyapunov function cannot exist and the condition that $A_1A_2^{-1}$ has no negative eigenvalue is necessary and sufficient for the existence of a copositive common quadratic Lyapunov function for for Σ_{A_1} and Σ_{A_2}.
(c) Suppose that $A_1A_2^{-1}$ has a negative eigenvalue; namely, $A_1 + \gamma A_2$ is non-Hurwitz for some $\gamma \in \mathbb{R}^+$. It now follows from Theorem 2.1 that there exists some switching sequence for which the associated switching system

$$\Sigma_S : \dot{x} = A(t)x \quad A(t) \in \{A_1, A_2\}$$

is not globally exponentially stable. Thus for this class of switching system CQLF existence is not a conservative way of establishing stability ; equivalently, Hurwitz stability of the pencil $A_1 + \gamma A_2$; $\forall \gamma \geq 0$ is necessary and sufficient for the associated switched linear system. □

A natural question arises in view of item (c); namely, does the Hurwitz stability of the polytope $CO(A_1, ..., A_k)$ (here as before the A_i are Metzler, Hurwitz and in $\mathbb{R}^{2\times 2}$ and $CO(\cdot)$ denotes convex hull) imply the stability of any associated switched system. The answer to this question is yes and follows from the following edge theorem for positive matrices.

Theorem 3.3: Consider the switched linear system constructed by switching between the Hurwitz Metzler matrices $\{A_1, .., A_k\}, k < \infty, A_i \in \mathbb{R}^{2\times 2}$. Then this system is globally uniformly asymptotically stable iff the switched system constructed by switching between any pair of matrices (A_i, A_j) is globally uniformly asymptotically stabble for all $1 \leq i < j \leq k$.

Proof : First , assume without loss of generality that for all $a > 0, 1 \leq i < j \leq k$ matrices $A_i - aA_j$ are not zero . Let R_+^2 be nonnegative orthant in R^2 .Consider an arbitrary nonzero vector $x \in R_+^2$ and associate with it a (convex) cone $C_{(x)}$ generated by $\{A_1x, .., A_kx\}$ and (convex) cones $C_{(x),i,j}$ generated by $\{A_ix, A_jx\}$, $1 \leq i < j \leq k$. Clearly , since we are in R^2 , $C_{(x)} = \cup_{1\leq i<j\leq k} C_{(x),i,j}$.

As the switched system $\{A_i, A_j\}$ is GUAS and the matrices are Metzler, $C_{(x),i,j} \cap R^2_+ = \{0\}$ for all $1 \le i < j \le k$ and nonzero $x \in R^2_+$. Therefore $C_{(x)} \cap R^2_+ = \{0\}$. For a nonzero vector $x \in R^2$ define $arg(x)$ in the usual way (viewing x as a complex number). It follows that $\frac{\pi}{2} < arg(A_i x) < \frac{3\pi}{4}$ for all $1 \le i \le k$ and nonzero $x \in R^2_+$. Let $(l(x), u(x)), 1 \le l(x), u(x) \le k$ be (nonunique) pair such that $arg(A_{l(x)}x) \le arg(A_i x) \le arg(A_{u(x)}x)$. Then clearly $C_{(x)} = C_{(x),l(x),u(x)}$. Define $D_{(i,j)} = \{y \in R^2_+, x \neq 0 : C_{(y)} = C_{(y),i,j}, arg(A_i(y)) \le arg(A_j(y))$.Here (i,j) is just of pair of integers , not necessarily ordered and possibly equal . It follows that $R^2_+ - \{0\} = \cup_{1 \le i,j \le k} D_{(i,j)}$. Now, it is clear that $D_{(i,j)}$ is a closed cone (without zero), not necessarily convex . Notice if $x \in D_{(i,j)}$ and $arg(A_i) < arg(A_m x) < arg(A_j), m \neq \{i,j\}$ then this nonzero vector x belongs to the interior of $D_{(i,j)}$. Consider $Symp = \{\hat{a} =: (a, 1-a)^T : 0 \le a \le 1\}$ and define $d_{(i,j)} = Symp \cap D_{(i,j)}$. We say that $\hat{a} < \hat{b}$ iff $a < b$. Notice that $d_{(i,j)}$ are closed and their (finite) union is equal to $Symp$. The only way for $x \in Symp$ not to be in the interior of some $d_{(i,j)}$ is that there exist $b > 0, 1 \le l \neq m \le k$ such that $A_l x = b A_m x$. As we assumed (without loss of generality) that for all $a > 0, 1 \le i < j \le k$ matrices $A_i - aA_j$ are not zero , thus there exists a finite subset $Sing =: \{0 \le \hat{a_1} < ... < \hat{a_q} \le 1\}$ such that all vectors in $Symp - Sing$ belong to interiors of some of $d_{(i,j)}$. By the standard connectivity argument we get that the closed 1-dimensional interval $symp$ has a finite partition (with common end points) into closed intervals , each of them belongs to some $d_{(i,j)}$. And this gives a partition of $R^2_+ - \{0\}$ into finite closed cones/wedges (with common border lines), each of them belongs to some $D_{(i,j)}$. As the switched system $\{A_{L(j)}, A_{U(j)}\}$ is GUAS for all $1 \le j \le m-1$ thus there exists (Hilbert) norms $< P_j x, x >^{\frac{1}{2}}, P \succ 0$ such that $< A_{L(j)}x, P_j x > \le 0$ and $< A_{U(j)}x, P_j x > \le 0$ for all $x \in R^2$. As $Sec_j \subset D_{(L(j),U(j))}$, we get that $< A_i x, P_j x > \le 0$ for all $x \in Sec_j$ and all $1 \le i \le k$. Let us choose large enough $T_1 = (0, y)^T, y > 0$ andconsider an arc of P_1-ellipsoid going trough T_1 , it will intersect the second ray r_2 at some point T_2 ; consider an arc of P_2-ellipsoid going trough T_2 , it will intersect the third ray r_2 at some point T_3 ; this process will end up at some point T_m at the x-axis . As a result we get a domain bounded by y-axis , chain of ellipsoidal arcs and y -axis . The main point is that this domain is invariant for the switched system $\{A_1, .., A_k\}$ and that the system has uniformly bounded trajectories . By a simple perturbation argument the same boundedness statement holds for switched systems constructed from $\{A_1 + \epsilon I, .., A_k + \epsilon I\}$ for some small enough positive ϵ. This last fact proves that the statement. □

IV. Higher dimensional systems

In view of the results obtained above for second order systems, and keeping in mind the restriction that the trajectories of a positive switched system are constrained to lie in the positive orthant for all time, it may seem reasonable to hope that analogous results could be obtained for higher dimensional systems. These thoughts would lead naturally to the conjecture that the Hurwitz-stability of the convex hull of a finite family of Metzler matrices in $\mathbb{R}^{n \times n}$ is both necessary and sufficient for the asymptotic stability of the associated switched system. Unfortunately, while this conjecture is as appealing as it is plausible, it is incorrect. This is shown by the following argument, first presented by Gurvits in [14]. For linear continuous time switching system Σ_S , where S is a compact set of bounded operators , define its **Joint Lyapunov Exponent** $JLE(S)$ as $\inf\{\lambda : \exists$ an equivalent norm $||.|| : ||exp(At)|| \le e^{\lambda t}, A \in S, t \ge 0\}$ Recall that an operator set S called irreducible if it does not have common nontrivial closed linear invariant subspace . In finite dimensional case it is not difficult to prove that for irreducible S the infimum above is attained , i.e. there exist an induced operator norm $||.||$ such that $||exp(At)|| \le e^{JLE(S)t}, A \in S, t \ge 0$ Essentially, there are three steps in the construction.

(i) Using the Lyapunov operator $X \to A^T X + XA$, two systems $\Sigma_{A_1}, \Sigma_{A_2}$ that leave a given pointed solid cone invariant (given by the positive semi-definite matrices in $\mathbb{R}^{2\times 2}$) are described such that (a) the convex hull of A_1, A_2 is Hurwitz-stable, and (b) the associated switched linear system is not asymptotically stable.

(ii) By approximating the cone of positive semi-definite matrices with a polyhedral cone, it is shown that there exist two systems leaving a *polyhedral* pointed solid cone invariant, with the same properties as in (i).

(iii) The final stage of the argument is to use an embedding result to show that the result obtained in (ii) implies that a similar example (possibly of much higher dimension) exists where the invariant cone can be taken to be the positive orthant. This then demonstrates the existence of two Metzler matrices whose convex hull is Hurwitz-stable, but for which the associated switched system is not asymptotically stable.

We begin the argument by considering the following two matrices

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & a \\ -b & o \end{pmatrix}$$

where $a > b \ge 0$. Then, for some $t_1, t_2 > 0$ the spectral radius $\rho((exp(A_1 t_1)(exp(A_2 t_2)) > 1$. Therefore, for some small $\epsilon < 0$ also $\rho((exp((A_1 + \epsilon I)t_1))(exp((A_2 + \epsilon I)t_2))) > 1$ (i.e. the associated switched system is unstable) and all matrices in the convex hull $CO(\{A_1 + \epsilon I, A_2 + \epsilon I\})$ are Hurwitz. It is clear that the **Joint Lyapunov Exponent** $JLE(S) > 0, S = \{A_1, A_2\}$ and this S is irreducible . Thus the switched system constructed from $\{A_1 - JLE(S)I, A_2 - JLE(S)I\}$ is not (GUAS). Let us now consider a polyhedral cone in R^N ,
$Cone(z_1, ..., z_k) =: \{x = \sum_{1 \le i \le k < \infty} \alpha_i z_i : \alpha_i \ge 0\}$.
Assume that there exists $h \in R^N$ such that $< h, z_i >= 1, 1 \le i \le k$. It follows that the polyhedral cone $Cone(z_1, ..., z_k)$ is invariant respect to $exp(At), t \ge 0$ iff there exists $\tau > 0$ such that $(I + \tau A)(Cone(z_1, ..., z_k)) \subset Cone(z_1, .., z_k)$.

The following argument is crucial to the rest of the paper. We consider a polyhedral cone $Cone(z_1, ..., z_k)$ with nonempty interior in R^N , $k, N < \infty$, such that there exists $h \in R^N$ satisfying $< h, z_i >= 1, 1 \le i \le k$. Define a linear operator $\Phi : R^k \to R^N$ by its actions on a canonical basis in R^k : $\Phi(e_i) = z_i, 1 \le i \le k$.

Then the following statements hold :

1) Suppose that $x_m = \sum_{1 \le i \le k < \infty} \alpha_{i,m} z_i, \alpha_{i,m} \ge 0$. Then $\lim_{m \to \infty} x_m = 0$ iff $\lim_{mm \to \infty} \sum_{1 \le i \le k} \alpha_{i,m} = 0$.

2) Suppose that (non-uniquely) $A(z_i) = \sum_{1\leq j\leq k<\infty} a_{i,j}z_j; a_{i,j} \geq 0$ if $i \neq j$. Now define a Metzler $k \times k$ matrix $A^{(M)} = (a_{i,j} : 1 \leq i,j \leq k)$. Then $\Phi A^{(M)} = A\Phi$ and $A^{(M)}$ is Hurwitz iff A is Hurwitz .
3) Consider the switched linear system constructed from the matrices $\{A_1, ..., A_l\}$ in $R^{N\times N}$ with invariant polyhedral cone $Cone(z_1, ..., z_k)$ and the corresponding Metzler embedding to R^k. Then all matrices in the convex hull $CO(\{A_1, ..., A_l\})$ are Hurwitz iff all matrices in the convex hull $CO(\{A_1^{(M)}, ..., A_l^{(M)}\})$ are Hurwitz . The switched systems associated with the sets of matrices, $(\{A_1, ..., A_l\})$, and $\{A_1^{(M)}, ..., A_l^{(M)}\}$, are equivalent.

The above remarks show that the main conjecture presented in this section would imply that the same statement holds for all switched linear systems having invariant polyhedral pointed cone with nonempty interior. In what follows we will present a counter-example to this equivalent "polyhedral" conjecture. As mentioned above, the idea is first to get a counter-example for a non-polyhedral cone and to approximate that non-polyhedral cone by a polyhedral one . Consider any two matrices A_1, A_2 such that $aA_1 + (1-a)A_2$ is Hurwitz for all $0 \leq a \leq 1$ but the **Joint Lyapunov Exponent** $JLE(\{A_1, A_2\}) > 0$. To be specific, let us consider the two 2×2 matrices already mentioned. We then associate with them "matrix" operators $\hat{A}_1(X) = A_1X + XA_1^*, \hat{A}_2(X) = A_2X + XA_2^*$ acting on a linear space over reals of all 2×2 Hermitian matrices . Let $SPD(2)$ be a pointed cone with nonempty interior consisting of all 2×2 positive semidefinite matrices . It is clear that $exp(\hat{A}_i t)(SPD(2)) \subset SPD(2); i = 1,2; t \geq 0$. It follows that for any $\epsilon > 0$ there exist $\tau > 0$ and two matrix operators $\Delta_i, i = 1,2$ such that $(\tau I + \hat{A}_i + \Delta_i)(SPD(2)) \subset SPD(2); ||\Delta_i|| \leq \epsilon; i = 1,2$. Using standard approximation of an arbitrary pointed finite dimensional cone by a polyhedral one , we get that for any $\epsilon > 0$ there exist a polyhedral cone with nonempty interior $PH_\epsilon \subset SPD(2)$ and two matrix operators $\delta_i, i = 1,2$ such that $(\tau I + \hat{A}_i + \Delta_i + \delta_i)(PH_\epsilon) \subset PH_\epsilon; ||\Delta_i||, ||\delta_i|| \leq \epsilon; i = 1,2$.

Let us put things together :

1) It is easy to see that operators $a\hat{A}_1 + (1-a)\hat{A}_2$ are Hurwitz for all $0 \leq a \leq 1$.
2) It is easy to see that $JLE(\{\hat{A}_1, \hat{A}_2\}) > 0$.
3) Thus we can choose small enough $\epsilon > 0$ such that for $B_{i,\epsilon} =: \hat{A}_i + \Delta_i + \delta_i; i = 1,2$ the following hold : operators $aB_{1,\epsilon} + (1-a)B_{2,\epsilon}$ are Hurwitz for all $0 \leq a \leq 1$;
$JLE(\{B_{1,\epsilon}, B_{2,\epsilon}\}) > 0$;$exp(B_{i,\epsilon}t)(PH_\epsilon) \subset PH_\epsilon$.
4) The last item gives the needed counterexample.

V. Compact sets of second order positive systems

In this section we consider switching systems

$$\Sigma_S : \dot{x} = A(t)x \quad A(t) \in S. \tag{4}$$

Here S is a compact set of 2×2 Metzler matrices . Our goal is to generalize Theorem 3.3 to this case . First , we recall [?] that there exist a 2-dimensional switching system Σ_S such that for all finite subsets $F \subset S$ switching system Σ_F is absolutely asimptoticaly stable but Σ_S is not .(The very similar example is presented also in [?] for a discrete time systems). But in this positive case Theorem 3.3 does hold for infinite compact sets . Let us scetch a proof :
First assume wlog that the system is irreducible , i.e. in this case there is no common eigenvector for all $A \in S$. Using compactness and similarly to the proof above of theorem 3.3 , it is enough to prove just boundness of trajectories . In irredicible case boundness is equivalent to the inequality $JLE(S) \leq 0$ (see [?] , citegul) . If $JLE(S) > 0$ then there exist $A_1, .., A_m \in S$ and $t_i > 0, 1 \leq i \leq m < \infty$ such that

$$\rho(exp(A_m t_m)...exp(A_1 t_1)) > 1$$

But this inequality contradicts to (finite) Theorem 3.3 . That is it .
Slightly modifying the above argument,we get that for any bounded set S of 2×2 Metzler matrices the following formula holds : $JLE(S) = sup_{A,B\in S}\ JLE(A,B)$ and $JLE(A,B)$ is the largest (real in this case) eigenvalue in the convex hull $CO(A,B)$. The last quantity (i.e. largest eigenvalue) is easy to compute in small fixed number of standard operations . All together it gives $O(|S|^2)$ algorithm to compute **Joint Lyapunov Exponent** $JLE(S)$, where $|S|$ is a cardinality of S .

VI. Conclusions

In this paper we have presented some preliminary results on the stability of switched positive systems. These results establish that determining the stability of positive systems is not equivalent to the Hurwitz stability of a convex combination of a set of matrices as was conjectured in [13]. While this conjecture is know known to be false, the counterexamples are rare, leading to the hope that it may be possible to extend the work reported here and to to derive related, straightforward conditions for CQLF existence in this case.

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